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Note

Dynamic and Quasi-Equilibrium Lagrangian MHD in 1-D*

1. Introduction

Quasi-equilibrium calculations of 1-D MHD problems continue to be of interest [1–3]. The basic principle used in these works is a two-step process: (1) solution of the diffusion processes in a rigid mesh, and (2) relaxation of the mesh allowing adiabatic changes. The present note discusses a procedure for including the inertia term in a smooth way which allows for all ranges of dynamic and quasi-steady behavior.

In pinch experiments such as the ZT-40 [4] there is an initial dynamic implosion phase which the pressure balance model does not describe well, followed by a long-term, quasi equilibrium phase which is well described by it. Previously it was impossible to run the whole problem smoothly through both regimes with precisely the same algorithm. A problem which is an even more severe test of the versatility of the algorithm and which does not become too involved in the detail of a particular experiment is the ideal equilibrium Z-pinch as discussed by Pease [5]. This problem will be discussed in the present note. More extensive calculations on the ZT-40 machine are in progress and will be published later in an article in which the emphasis will be on the physics of the device.

2. MHD Equation

Since the diffusive effects are calculated in a fairly standard way, the following discussion is concerned with the adiabatic step. The Lagrangian variable x is taken to be proportional to the initial volume inside the radius r. Thus, the Eulerian position $r(x, t)$ is given in terms of the inverse compression $V(x, t)$ by

$$r\frac{\partial r}{\partial x} = V. \tag{1}$$

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In terms of the Lagrangian coordinate system the momentum equation is written as

$$\frac{du}{dt}+\frac{r}{\rho_0}\frac{\partial}{\partial x}\left(p+\frac{B_z^2}{2\mu_0}\right)+\frac{1}{\rho_0 r}\frac{\partial}{\partial x}\left(\frac{r^2B_\Theta^2}{2\mu_0}\right)=0. \tag{2}$$

The variables x and r are evaluated at cell edges labeled by the index i. The thermodynamic and magnetic properties are evaluated at the cell centers indicated by half-integer indices. The difference analog of (2) is then written as

$$\begin{aligned}&\frac{K(u^{n-1/2}-u^{n-3/2})}{\Delta t}\\&\quad+\frac{r_i^n}{\bar\rho_{0i}\bar{\Delta}x_i}\left[p_{i+1/2}^n-p_{i-1/2}^n+K(q_{i+1/2}^{n-1/2}-q_{i-1/2}^{n-1/2})+\frac{(B_{z,i+1/2}^n)^2-(B_{z,i-1/2}^n)^2}{2\mu_0}\right]\\&\quad+\frac{1}{\bar\rho_{0,i}r_i^n\overline{\Delta x_i}}\left[\frac{(r_{i+1/2}^nB_{\Theta,i+1/2}^n)^2-(r_{i-1/2}^nB_{\Theta,i-1/2}^n)^2}{2\mu_0}\right]=0,\end{aligned} \tag{3}$$

where the artificial viscosity $q_{i+1/2}^{n-1/2}$ is given by [6]

$$q_{i+1/2}^{n-1/2}=\begin{cases}\rho_{0,i+1/2}c^2\,|u_i^{n-1/2}-u_{i+1}^{n-1/2}|^2, & u_i^n\geqslant u_{i+1}^n,\\ 0, & u_i^n\leqslant u_{i+1}^n,\end{cases} \tag{4}$$

and c is a constant of order unity. The $\bar\rho_{0,i}$ and $\bar{\Delta}x_i$ are normally located at cell centers, but the barred quantities are averaged for the cell edges. To force a quasi-steady pressure balance calculation the constant K is set equal to zero. For the dynamic problem K is set equal to unity. The time index n indicates the advanced time. The difference expression for the velocity $u_i^{n-1/2}$ is

$$u_i^{n-1/2}=\frac{r^n-r^{n-1}}{\Delta t}. \tag{5}$$

Solution of (3) can be performed either before or after the diffusive calculation. The solution to (3) must be performed in conjunction with the energy and magnetic equations truncated by deletion of the diffusive terms. The truncated energy equation is

$$T_{i+1/2}^n-T_{i+1/2}^{n-1}=-\frac{T_{i+1/2}^n(p_T)_{i+1/2}^n}{(e_T)_{i+1/2}^n\rho_{0,i+1/2}}(V_{i+1/2}^n-V_{i+1/2}^{n-1}), \tag{6}$$

where e is the specific internal energy and $V_{i+1/2}^n$ is the inverse compression,

$$V_{i+1/2}^n=\frac{(r_{i+1}^n)^2-(r_i^n)^2}{2\,\Delta x_{i+1/2}}, \tag{7}$$

and e_T and p_T are temperature derivatives. The magnetic equations are

$$\frac{V^n_{i+1/2}B^n_{\Theta,i+1/2}}{r^n_{i+1/2}} = \frac{V^{n-1}_{i+1/2}B^{n-1}_{\Theta,i+1/2}}{r^{n-1}_{i+1/2}}, \tag{8}$$

$$V^n_{i+1/2}B^n_{z,i+1/2} = V^{n-1}_{i+1/2}B^{n-1}_{z,i+1/2}. \tag{9}$$

By the use of (4)–(9), (3) can be put in the form of a nonlinear system of equations for the mesh positions of the form

$$f_i(r^n_{i-1}, r^n_i, r^n_{i+1}) = 0. \tag{10}$$

This system is solved by the Newton–Raphson procedure with the residuals Δr^n_i obtained by a solution of the tri-diagonal system,

$$-f_i = \frac{\partial f_i}{\partial r^n_{i-1}}\Delta r^n_{i-1} + \frac{\partial f_i}{\partial r^n_i}\Delta r^n_i + \frac{\partial f_i}{\partial r^n_{i+1}}\Delta r^n_{i+1}. \tag{11}$$

The actual expressions for the derivatives in (11) turn out to be

$$\begin{aligned}\frac{\partial f_i}{\partial r^n_i} = {} & \frac{K}{(\Delta t)^2} - \frac{(r^n_{i+1/2})^2}{(\bar\rho_0\bar\Delta x)_i}\left[\left(\frac{p_T T_v}{\Delta x}\right)^n_{i+1/2} + \left(\frac{p_T T_v}{\Delta x}\right)^n_{i-1/2}\right] \\ & + K\left\{\frac{q^{n-1/2}_{i+1/2} - q^{n-1/2}_{i-1/2}}{(\bar\rho_0\bar\Delta x)_i} + \frac{2c^2 r^n_i}{\Delta t(\bar\rho_0\bar\Delta x)_i}\left[\left(\frac{\rho_0}{V}\right)^n_{i+1/2} + \left(\frac{\rho_0}{V}\right)^n_{i-1/2}\right]\right\} \\ & + \frac{1}{2\mu_0(\bar\rho_0\bar\Delta x)_i}\left\{(B^n_{i+1/2})^2 - (B^n_{i-1/2})^2 + 2(r^n_i)^2\left[\left(\frac{B^2_z}{V\Delta x}\right)^n_{i+1/2} + \left(\frac{B^2_z}{V\Delta x}\right)^n_{i-1/2}\right]\right\} \\ & + \frac{1}{2\mu_0(\bar\rho_0\bar\Delta x)_i}\left\{-\frac{1}{(r^n_i)^2}\left[(\bar r^2 B^2_\Theta)^n_{i+1/2} - (\bar r^2 B^2_\Theta)^n_{i-1/2}\right]\right. \\ & \left. + \frac{4}{(r^n_i)}\left[\left(\frac{\bar r^2 B^2_\Theta}{\Delta x}\right) + \left(\frac{\bar r^2 B^2_\Theta}{\Delta x}\right)\right] - 2\left[\left(\frac{\bar r^2 B^2_\Theta}{V\Delta x}\right)^n_{i+1/2} + \left(\frac{\bar r^2 B^2_\Theta}{V\Delta x}\right)^n_{i+1/2}\right]\right\}, \end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial f_i}{\partial r^n_{i+1}} = {} & \frac{r^n_i r^n_{i+1}}{(\bar\rho_0\bar\Delta x)_i}\left(\frac{p_T T_v}{\Delta x}\right)_{i+1/2} - K\frac{2c^2 r^n_i}{\Delta t^2(\bar\rho_0\bar\Delta_x)_i}\left(\frac{\rho_0}{V}\right)^n_{i+1/2} \\ & - \frac{1}{\mu_0(\bar\rho_0\bar\Delta x)_i} r^n_i r^n_{i+1}\left(\frac{B^2_z}{V\Delta x}\right)^n_{i+1/2} \\ & + \frac{2}{\mu_0(\bar\rho_0\bar\Delta x r)^n_i}\left[r^n_{i+1}\left(\frac{\bar r^2 B^2_\Theta}{V\Delta x}\right)^n_{i+1/2} - 2\left(\frac{\bar r^2 B^2_\Theta}{\Delta x}\right)^n_{i+1/2}\right], \end{aligned} \tag{13}$$

$$\frac{\partial f_i}{\partial r_{i-1}^n} = \frac{r_i^n r_{i-1}^n}{(\bar{\rho}_0 \bar{\Delta x})_i} \left(\frac{p_T T_v}{\Delta x} \right)^n_{i-1/2} - K \frac{2c^2 r_i^n}{\Delta t (\bar{\rho}_0 \bar{\Delta x})_i} \left(\frac{\rho_0}{V} \right)^n_{i-1/2}$$
$$- \frac{1}{\mu_0 (\bar{\rho}_0 \bar{\Delta x})_i} r_i^n r_{i-1}^n \left(\frac{B_z^2}{V \Delta x} \right)^n_{i-1/2}$$
$$+ \frac{2}{\mu_0 (\bar{\rho}_0 \bar{\Delta x} r)_i^n} \left[r_{i-1}^n \left(\frac{\bar{r}^2 B_\Theta^2}{V \Delta x} \right)^n_{i-1/2} - 2 \left(\frac{\bar{r}^2 B_\Theta^2}{V \Delta x} \right)^n_{i-1/2} \right]. \tag{14}$$

The thermodynamic and magnetic quantities in (12)–(14) are calculated through (6)–(9). The iterations are judged to be converged when the residuals in (11) remain unchanged to within a specified accuracy on successive iterations.

At this point it is worth comparing the present method with other methods. Lindemuth and Killeen [7] used the Newton–Raphson method to get convergence of their coupled, non-linear equations. However, they were treating 2-D Eulerian equations without splitting off the adiabatic part. Thus their algorithm is much more involved. Harlow and Amsden [8] use truncated equations to obtain pressure by a point–iterative procedure in Eulerian hydrodynamics. Although the splitting method is analogous, it is found that a global relaxation procedure is necessary for convergence in the present model for Lagrangian MHD.

3. The Equilibrium Z-Pinch

In the original equilibrium Z-pinch model [5], the external radius was assumed fixed and pressure balance was assumed. In the present test calculation the pressure balance relation is replaced by the momentum equation, as discussed in the previous section. An initial, uniform temperature is given and the time-dependent motion is followed until the long-term thermal balance is reached. In order to keep the outer radius at a fixed value the driving current must be programmed properly. This requires a special control program which is not used in other calculations and which is illustrated in Fig. 1.

The calculated current I_c is obtained as follows:

(1) Hold the outer radius b fixed.

(2) Guess a value for the driving current $I_g(t + \Delta t)$.

(3) Solve the momentum equation for the new configuration at time $t + \Delta t$.

(4) Calculate the new current $I_c(t + \Delta t)$ from the generalized Bennett relation,

$$\mu_0 I^2 = 16\pi N \kappa \bar{T} - 8\pi^2 K \int_0^b r^2 p \frac{du}{dr} dr, \tag{15}$$

where N is the line density and $\bar{T}$ is the average temperature. In the forced equilibrium calculation K is set equal to zero and the usual Bennett relation is used. Experience indicates that if the relaxation parameter ω indicated in Fig. 1 is taken as

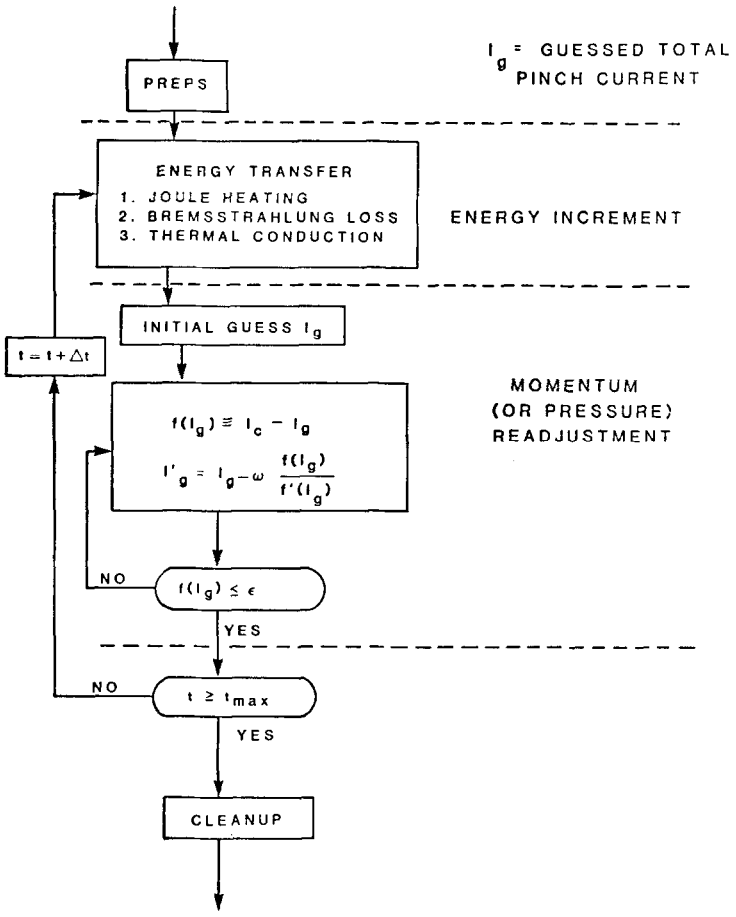

FIG. 1. Flow diagram for the programmed current iteration.

unity, the iterations do not converge. However, when a smaller value, $\omega \approx 0.2$, is taken, the convergence is quite dependable. It is worth noting that the only difference between the dynamic and forced equilibrium calculations is the setting of K to unity or to zero.

It is important to consider the initial conditions in terms of the equations given. The initial configuration is taken to be a plasma of uniform density and temperature. In the dynamic calculation the plasma moves in the first time step exactly according to the above equations with the current set at $t + \Delta t$ to the value required to maintain a fixed outer radius b. In the case of the forced equilibrium calculation, the initial configuration is, of course, not one of pressure balance for confinement by a current. However, the state at $t + \Delta t$ is the pressure balance state connected to it by the appropriate adiabatic adjustment. From then on, each configuration satisfies pressure balance exactly.

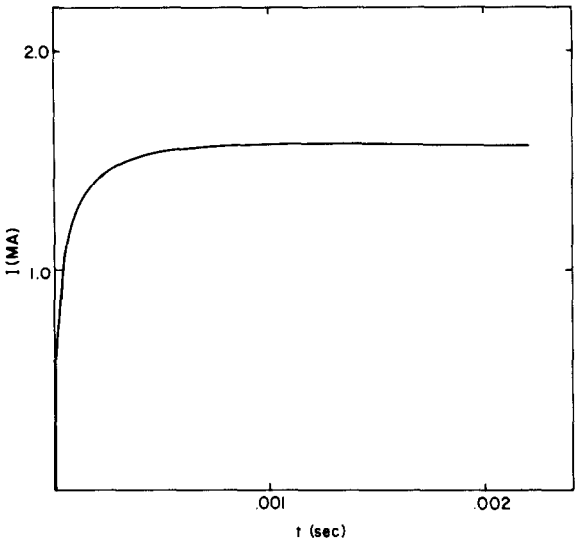

FIG. 2. Z-Pinch plasma current vs time over a long time scale.

In the particular example considered here, the plasma was started at 10 eV and the calculation run until the Pease steady state was established. This took about 3 msec in physical time. This calculation was a good test of how the method goes smoothly from a highly dynamic phase to a quasi-steady phase. In the early stages of the problem $\Delta t \sim 10^{-14}$ was required to follow the dynamics brought about by the extremely rapid current rise. As the inertia term ceased to be important, the calculation went smoothly into a quasi-equilibrium with the constant K remaining unity. The current vs time is plotted in Fig. 2. The whole calculation took about 6 min of 7600 time. A forced quasi-equilibrium calculation with $K = 0$ took about 3 min of 7600 time, but did an inadequate job of following the detailed variation of velocity and temperature profiles in the early stages.

From the point of view of following the plasma to full Pease thermal balance, the forced equilibrium calculation is quite adequate in terms of the form the current rise vs time taken. Indeed, there is no observable discrepancy between the $K = 0$ and $K = 1$ cases. In the dynamic case, the details of the early motion are quite different from those found in the pressure balance case, but to plot all the details here is not as important as to make the main point that the dynamic algorithm is such that these details can be followed when they are important. Full advantage will be taken of this property in work on the above mentioned ZT-40 calculations in progress.

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